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A METHOD OF STUDYING THE STABILITY OF AUTONOMOUS SYSTEMS

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When Liapunov's direct method is used to study the stability of nonlinear systems and attempts are made to construct a Liapunov function with a derivative of constant sign or sign-definite, serious difficulties often occur. In the present paper a method is proposed for studying the stability of autonomous systems wherein use is made of an auxiliary function $V(\mathbf{x})$. The method is not connected with the conditions for $V(\mathbf{x})$ and its derivative with respect to time to be of constant sign or sign-definite. Instead, the function $V(\mathbf{x})$ along the trajectories of the system under study is required to satisfy a second order linear differential equation and certain boundary conditions. A theorem for the existence of the function $V(\mathbf{x})$ is proved and an effective method is given for constructing it is the solution of a Dirichlet problem for a degenerate elliptic operator of a special type; this makes it possible to obtain $V(\mathbf{x})$ numerically with the help of a computer. The function $V(\mathbf{x})$ can be used, not only for the study of stability, but also to determine regions of attraction and to obtain the invariant sets of autonomous systems, in particular, the limit cycles of second order systems.

1. We consider the system of equations of a perturbed motion

$$\mathbf{x}^{\cdot} = f(\mathbf{x}) \tag{1.1}$$

defined in some bounded domain $D \subset R^m$ and such that $f(\mathbf{x}) \in C^{(1)}(D)$. Here, and in what follows, by $C^{(k)}(D)$ we shall mean the space of functions which have in D continuous partial derivatives to order k inclusive, and by $C^{(k+\alpha)}(D)$ we shall mean the space of functions which have in D partial derivatives of order k which satisfy a Hölder condition with exponent $0 < \alpha < 1$. Let $\Omega = \{\mathbf{x} : \|\mathbf{x}\| \leqslant r\} \subset D$, and let Σ be the boundary of Ω. The intrinsic norm in R^m will be denoted by $\|\cdot\|$.

We introduce now an auxiliary system of equations for the perturbed motion

$$\mathbf{x}^{\cdot} = h(\mathbf{x}) \tag{1.2}$$

where

$$h(\mathbf{x}) = \varphi_1(\mathbf{x}) f(\mathbf{x}) + \varphi_2(\mathbf{x}) \mathbf{x} \tag{1.3}$$

Here $\varphi_1(\mathbf{x})$ and $\varphi_2(\mathbf{x})$ are scalar matrices, their diagonal elements being defined as follows:

$$\varphi_1^{ii}(\mathbf{x}) = \begin{cases} 1, & \|\mathbf{x}\| \leqslant r - \xi \\ \exp\left[-\left(\frac{\|\mathbf{x}\| - r + \xi}{\|\mathbf{x}\| - r + {}^1/_2\xi}\right)^6\right], & r - \xi < \|\mathbf{x}\| < r - \frac{1}{2}\xi \\ 0, & r - \frac{1}{2}\xi \leqslant \|\mathbf{x}\| \end{cases}$$

$$\varphi_2^{ii}(\mathbf{x}) = \begin{cases} 0, & \|\mathbf{x}\| \leqslant r - \xi \\ \exp\left[-\left(\frac{\|\mathbf{x}\| - r}{\|\mathbf{x}\| - r + \xi}\right)^6\right], & r - \xi < \|\mathbf{x}\| < r \\ 1, & r \leqslant \|\mathbf{x}\| \end{cases}$$

where $i = 1, 2, \ldots, m$, and ξ is a sufficiently small positive number.

Lemma 1.1. If the trivial solution of the system (1.2) is stable, asymptotically stable, or unstable, then the trivial solution of the system (1.1) is, respectively, stable, asymptotically stable, or unstable.

Proof. By virtue of the choice of functions $\varphi_1(\mathbf{x})$ and $\varphi_2(\mathbf{x})$ in the sphere of radius $r - \xi$, we have $f(\mathbf{x}) \equiv h(\mathbf{x})$. The assertion of the lemma is therefore an immediate consequence of the coincidence of the trajectories of the systems (1.1) and (1.2) in the sphere $\|\mathbf{x}\| < r - \xi$ and of the definitions of stability, asymptotic stability and instability in the sense of Liapunov [1, 2].

Let us assume that in $\Omega \bigcup \Sigma$ there exists a function $V(\mathbf{x}) \in C^{(2)}(\Omega)$ possessing the following properties (k and g are constants): (A) $V(0) = 0$; (B) $V^{\cdot\cdot}(\mathbf{x}) = kV(\mathbf{x})$, $(k > 0)$; (C) $V(\mathbf{x}) = g$ $(g > 0,\ \mathbf{x} \in \Sigma)$; and (D) $|V(\mathbf{x})| \leqslant g$, if $\mathbf{x} \in \Omega$.

Here, and in what follows, we shall understand $V^{\cdot}(\mathbf{x})$ to mean the first total derivative of the function $V(x)$ with respect to time at the point $\mathbf{x}$ by virtue of the system (1.2), and by $V^{\cdot\cdot}(\mathbf{x})$ we shall mean the second total derivative. We introduce the following notation:

$$\mathbf{x}_0 = \mathbf{x}(t_0), \quad V_0 = V(\mathbf{x}_0), \quad V_0^{\cdot} = V^{\cdot}(\mathbf{x}_0), \qquad \lambda = +k^{-1/2}$$

$$P_\pm(\mathbf{x}) = {}^1/_2\,[V(\mathbf{x}) \pm \lambda V^{\cdot}(\mathbf{x})]$$

$$H_1 = \{\mathbf{x}: P_+(\mathbf{x}) > 0\}, \quad H_2 = \{\mathbf{x}: P_+(\mathbf{x}) = 0\} \bigcap \{\mathbf{x}: P_-(\mathbf{x}) \neq 0\}$$

$$H_3 = \{\mathbf{x}: P_+(\mathbf{x}) = 0\} \bigcap \{\mathbf{x}: P_-(\mathbf{x}) = 0\}, \quad H_4 = \{\mathbf{x}: P_+(\mathbf{x}) < 0\}$$

Lemma 1.2. 1) If $\mathbf{x}_0 \in H_1$, the trajectory $\mathbf{x}(t)$ with the initial condition $\mathbf{x}_0$ reaches the boundary of the domain Ω in a finite time;

2) If $\mathbf{x}_0 \in H_2$, then $\mathbf{x}(t) \in \Omega$ for $t > t_0$ and $\lim_{t\to\infty} \mathbf{x}(t) \in H_3$;

3) If $\mathbf{x}_0 \in H_3$, then $\mathbf{x}(t) \in H_3$ for $t > t_0$;

4) The set H_4 is empty.

(By the expression $\lim_{t\to\infty} \mathbf{x}(t) \in H_3$ we mean the following: for arbitrary $\varepsilon > 0$ we can find a $T(\varepsilon) > t_0$ such that for all $t > T(\varepsilon)$ we shall have $\rho(\mathbf{x}(t), H_3) < \varepsilon$.)

Proof. A solution of the equation $V^{\cdot\cdot}(\mathbf{x}) = kV(\mathbf{x})$ along the trajectory $\mathbf{x}(t)$ has the form

$$V(t) = V(\mathbf{x}(t)) = P_-(\mathbf{x}_0)\exp\left(-\frac{t - t_0}{\lambda}\right) + P_+(\mathbf{x}_0)\exp\left(\frac{t - t_0}{\lambda}\right) \tag{1.4}$$

Let $x_0 \in H_1$. Then $P_+(x_0) > 0$ and $\lim_{t\to\infty} V(t) = +\infty$. Therefore, by Property D, we can find a $T > t_0$ such that $x(T) \in \Sigma$. Let $x_0 \in H_2$. Then $P_+(x_0) = 0$ and $P_-(x_0) = V_0$. Therefore, $V(t) = V_0 \exp[-(t - t_0)/\lambda]$ and, by Properties C and D, $x(t) \in \Omega$ for $t > t_0$. Obviously, $\lim_{t\to\infty} V(t) = \lim_{t\to\infty} V^{\cdot}(t) = 0$, and hence, we have $\lim_{t\to\infty} P_+(x(t)) = \lim_{t\to\infty} P_-(x(t)) = 0$. Therefore, by virtue of the continuity of the functions $P_+(x)$ and $P_-(x)$, we have $\lim_{t\to\infty} x(t) \in H_3$. Let $x_0 \in H_3$. Then $P_+(x_0) = P_-(x_0) = 0$ and $V(t) = V^{\cdot}(t) = 0$ for $t \geqslant t_0$, by virtue of (1.4). It then follows that for $t > t_0$, $P_+(x(t)) = P_-(x(t)) = 0$ and $x(t) \in H_3$ for $t > t_0$.

Let $x_0 \in H_4$. It is readily seen that if $P_-(x_0) < 0$, then at some instant t_1 the function $V(t)$ attains a maximum equal to $-\sqrt{V_0^2 - \lambda^2 V_0^{\cdot 2}} \geqslant -g$, by virtue of Property D. By virtue of Property C, $x(t_1) \in \Omega$. Since it follows from (1.4) that $V(t)$ can have only one extremum, we conclude that the function $V(t)$ is monotonically decreasing on the semi-axis $[t_1, +\infty)$. But if $P_-(x_0) \geqslant 0$, then $V(t)$ decreases monotonically on the semi-axis $[t_0, +\infty)$. In both cases, by virtue of the Properties C and D, we have $x(t) \in \Omega$ for $t > t_0$, and consequently, $|V(x(t))| \leqslant g$ for $t > t_0$. However, in the case $x_0 \in H_4$, it follows from (1.4) that $\lim_{t\to\infty} V(t) = -\infty$. The resulting contradiction is a consequence of the assumption that the set H_4 is not empty.

Theorem 1.1. If a $\delta > 0$ can be found such that the set $\{x : 0 < \|x\| < \delta\} \subset H_2$, then the trivial solution of the system (1.2) is asymptotically stable.

Proof. From the definition of the set H_2, the Property A, and the obvious equality $h(0) = 0$, it follows that in the sphere $\|x\| < \delta$ the functions $V(x)$ and $V^{\cdot}(x)$ are of sign-definite, whereupon the inequality $V(x)V^{\cdot}(x) < 0$ is satisfied for $x \neq 0$. Thus, in this sphere the function $V(x)$ is a Liapunov function satisfying the conditions of the theorem concerning asymptotic stability in the case of a steady motion (see [2]).

Theorem 1.2. If $x = 0$ is the unique limit point of the set H_2 belonging to the set H_3, then the set H_2 is a domain of attraction for the trivial solution of system (1.2).

Proof. If the conditions of the theorem are satisfied, then the distance $\rho(H_2, H_3 \setminus 0) > 0$. Therefore, by virtue of Lemma 1.2, an arbitrary trajectory $x(t)$, beginning in the set H_2, can only tend towards the origin, and $\lim_{t\to\infty} x(t) = 0$.

Consider now two sequences of positive numbers $\{r_i\}$ and $\{\delta_j\}$, tending to zero. We denote $\Omega_i = \{x : \|x\| < r_i\}$. Let us assume that in each domain Ω_i a function $V_i(x) \in C^{(2)}(\Omega_i)$ exists and possesses in this domain the Properties A through D. In the domain Ω_i we select sets H_1^i, H_2^i, H_3^i.

Theorem 1.3. If for some r_i we can find a $\delta_j < r_i$, such that the set $\{x : \|x\| < \delta_j\} \subset H_2^i \cup H_3^i$, then there exist in the domain Ω_i bounded solutions of the system (1.2), which are distinct from the trivial solution.

Proof. By Property C and the definition of the set $H_3{}^i$ the distance $\rho(H_3^i, \Sigma_i) > 0$. Therefore, an arbitrary trajectory, starting from the sphere $\|x\| < \delta_j$, remains, as a consequence of Lemma 1.2, in the sphere $\|x\| < r_i$ for all $t > t_0$.

The following obvious theorem is a consequence of Theorem 1.3.

Theorem 1.4. If the conditions of Theorem 1.3 are satisfied for arbitrary r_i from some value onward, it follows that the trivial solution of the system (1.2) is stable.

Theorem 1.5. If for some r_i the origin is a limit point of the set H_1^i, then the trivial solution of the system (1.2) is unstable.

Proof. By virtue of Lemma 1.2 and the definition of a limit point, we can find, in an arbitrarily small neighborhood of the origin, a point through which the trajectory $\mathbf{x}(t)$ passes and reaches a sphere of radius r_i in a finite time; this corresponds to the instability in Liapunov sense.

Theorem 1.5 is a particular case of a theorem of Matrosov (see [3], Theorem 3.2). The set H_3 is, as a consequence of Lemma 1.2, an invariant set (see [4]) of the system (1.2). With the help of the function $V(\mathbf{x})$ we can construct a topological picture of the distribution of the set H_3 in Ω; this sometimes enables us to obtain information on the behavior of the trajectories of the system (1.2). Thus, for example, if for $m = 2$ there is in the domain Ω a connected component K of the set H_3, topologically equivalent to a circle and consisting of ordinary points of the system (1.2), then K may turn out to be a limit cycle. The possibility of using the function $V(\mathbf{x})$ to investigate systems of differential equations qualitatively was first pointed out by Nemytskii [5].

2. We now prove the existence of a function $V(\mathbf{x}) \in C^{(2)}(\Omega)$ and possessing the Properties A through D. Following [6, 7], we denote

$$a^{ij}(\mathbf{x}) = h_i(\mathbf{x})h_j(\mathbf{x}), \quad b^i(\mathbf{x}) = \sum_{j=1}^{m} \frac{\partial h_i(\mathbf{x})}{\partial x_j} h_j(\mathbf{x})$$

$$c = -k, \quad V_{x_i} = \partial V(\mathbf{x}) / \partial x_i, \quad V_{x_i x_j} = \partial^2 V(\mathbf{x}) / \partial x_i \partial x_j$$

with the undestanding that repeated indices i and j represent summations from 1 to m. Property B can then be written as follows:

$$L(V) \equiv a^{ij} V_{x_i x_j} + b^i V_{x_i} + cV = 0 \tag{2.1}$$

where L is a second order linear differential operator. All principal minors of the matrix

$$\| a^{ij}(\mathbf{x}) \|_1^m = \begin{Vmatrix} h_1^2(\mathbf{x}) & h_1(\mathbf{x})h_2(\mathbf{x}) & \ldots & h_1(\mathbf{x})h_m(\mathbf{x}) \\ h_2(\mathbf{x})h_1(\mathbf{x}) & h_2^2(\mathbf{x}) & \ldots & h_2(\mathbf{x})h_m(\mathbf{x}) \\ \ldots & \ldots & \ldots & \ldots \\ h_m(\mathbf{x})h_1(\mathbf{x}) & h_m(\mathbf{x})h_2(\mathbf{x}) & \ldots & h_m^2(\mathbf{x}) \end{Vmatrix}$$

are nonnegative in Ω; therefore, the quadratic form $a^{ij} z_i z_j$ is nonnegative in Ω [8] and the operator L is a degenerate elliptic operator in Ω [6, 7].

Associating the Property C with (2.1), we obtain the first boundary value problem for the operator L

$$L(V) = 0, \quad \mathbf{x} \in \Omega; \quad V(\mathbf{x}) = g, \quad \mathbf{x} \in \Sigma \tag{2.2}$$

For the proof of the existence of a solution $V(\mathbf{x})$ of the boundary value problem (2.2) and a study of its properties, we introduce the following additional notation, adopted in [6, 7]:

$$n(\mathbf{x}) = (n_1, n_2, \ldots, n_m)$$

$$\Sigma^\circ = \{\mathbf{x}\colon a^{ij} n_i n_j = 0\}, \quad b(\mathbf{x}) = (b^i - a_{x_j}^{ij}) n_i \quad (\mathbf{x} \in \Sigma^\circ)$$

$$\Sigma_0 = \{\mathbf{x}\colon b(\mathbf{x}) = 0\}, \quad \Sigma_1 = \{\mathbf{x}\colon b(\mathbf{x}) > 0\}$$

$$\Sigma_2 = \{\mathbf{x}\colon b(\mathbf{x}) < 0\}, \quad \Sigma_3 = \Sigma \setminus \Sigma^\circ$$

$$G = \overline{(\Sigma_0 \cup \Sigma_2)} \setminus (\Sigma_0 \cup \Sigma_2)$$

$$E = \{\mathbf{x}\colon \det \| a^{ij}(\mathbf{x}) \|_1^m = 0\} \cap (\Omega \cup \Sigma)$$

$$\eta^p \equiv c - \frac{1}{p} b^i_{x_i} + \frac{1}{p} a^{ij}_{x_i x_j}, \quad p > 1$$
$$B_{\bar{S}_l} \equiv c + \beta(m; l; b^i_{x_i}; \quad a^{ij}_{x_i x_j})$$
$$\bar{S}_l = (s_1, s_2, \ldots, s_l), \quad s_i = 1, 2, \ldots, m, \quad l \geqslant 1$$

where n (x) is the inner normal vector to Σ. An expanded expression for $B_{\bar{S}_l}$ was given in [6, 7]. Let C_k (Ω) denote the space of functions which have in Ω bounded generalized derivatives to order k inclusive, and let $W_p^{(l)}$ be a Sobolev space [9]. Let

$$L^*(V) = a^{ij} V_{x_i x_j} + b^{*i} V_{x_i} + c^* V$$
$$b^{*i} = 2a^{ij}_{x_i} - b^i, \quad c^* = a^{ij}_{x_i x_j} - b^i_{x_i} + c$$

where L^* is the operator conjugate to L. Obviously, for the case in question,

$$\Sigma_3 = \Sigma, \quad \Sigma_0 = \Sigma_1 = \Sigma_2 = \Sigma^\circ = G = \phi, \quad E = \Omega \cup \Sigma$$

Definition. We say that a function V (x), bounded and measurable in Ω, is a generalized solution of the boundary value problem

$$L(V) = \Phi(\mathbf{x}), \ \mathbf{x} \in \Omega; \quad V(\mathbf{x}) = \gamma(\mathbf{x}), \ \mathbf{x} \in \Sigma_2 \cup \Sigma_3$$

if for an arbitrary function $\Theta \in C^{(2)}(\Omega \cup \Sigma)$ and equal to zero on $\Sigma_1 \cup \Sigma_3$ the following integral identity is satisfied:

$$\int_\Omega V L^*(\Theta)\, d\mathbf{x} = \int_\Omega \Theta \Phi\, d\mathbf{x} - \int_{\Sigma_3} \gamma \frac{\partial \Theta}{\partial \nu} d\sigma + \int_{\Sigma_2} b\gamma\Theta\, d\sigma \tag{2.3}$$

$$\partial / \partial\nu \equiv a^{ij} \cos(n, x_i)\, \partial / \partial x_i$$

where Φ and γ are bounded measurable functions and $d\sigma$ is an element of area of Σ [6, 7]. For the case considered here the formula (2.3) takes the form

$$\int_\Omega V L^*(\Theta)\, d\mathbf{x} = -g \int_\Sigma \frac{d\Theta}{\partial \nu} d\sigma \tag{2.4}$$

Theorem 2.1. If the following conditions are satisfied in D: (1) $f(\mathbf{x}) \in C^{(4+\alpha)}(D)$; (2) the constant c is sufficiently large in absolute value so that the inequalities $\eta^2 < 0$, $c^* < 0$ and $B_{\bar{S}_3} < 0$ hold; then a function $V(\mathbf{x}) \in C^{(2)}(\Omega)$ exists and possesses the Properties A through D.

Proof. It was noted above that $c < 0$ and G has measure zero on Σ. In addition, it follows from the formulation of the boundary value problem (2.2) that $g = \text{const}$ on Σ_3. Therefore, by virtue of a theorem proved in [6] (Theorem 1), there exists in Ω a generalized solution V (x) of the boundary value problem (2.2), satisfying the maximum principle $|V(\mathbf{x})| \leqslant g$. This proves the Property D.

We prove now that the solution V (x) is unique. Let V_1 (x) be a solution of the problem (2.2) distinct from V (x). We denote $\psi(\mathbf{x}) = V(\mathbf{x}) - V_1(\mathbf{x})$, and consider the boundary value problem

$$L(\psi) = 0, \ \mathbf{x} \in \Omega; \quad \psi(\mathbf{x}) = 0, \ \mathbf{x} \in \Sigma \tag{2.5}$$

As a consequence of Theorem 2 in [6], there exists in Ω a generalized solution ψ (x) of the problem (2.5) belonging to $L_2(\Omega)$. From Eq. (2.4) we have

$$\int_\Omega \psi L^*(\Theta)\,d\mathbf{x} = 0 \tag{2.6}$$

Since $c^* < 0$ and the condition (2.6) is satisfied, the boundary value problem (2.5) satisfies the conditions of Theorem 3 of [6], by virtue of which $\psi(\mathbf{x}) = 0$ almost everywhere in Ω (*). Uniqueness of the solution of $V(\mathbf{x})$ then follows from this.

It remains to prove smoothness of the function $V(\mathbf{x})$. For this we introduce the auxiliary function $U(\mathbf{x}) = V(\mathbf{x}) - g$, and we consider the boundary value problem

$$L(U) = cg, \quad \mathbf{x} \in \Omega;\ U(\mathbf{x}) = 0, \ \mathbf{x} \in \Sigma \tag{2.7}$$

We continue the coefficient c of the operator L as a constant from Ω onto D. From the condition 1 of Theorem 2.1 and formula (1.3) it follows that $h(\mathbf{x}) \in C^{(4)}(D)$. Therefore, $a^{ij}(\mathbf{x}),\ b^i(\mathbf{x}) \in C^{(3)}(D)$, and, by virtue of the properties of the generalized derivatives, $a^{ij}(\mathbf{x}),\ b^i(\mathbf{x}) \in C_3(D)$ [9]. In addition $B_{\bar{S}_3} < 0$. Then the generalized solution $U(\mathbf{x})$ of problem (2.7) and, consequently, of $V(\mathbf{x})$ belong to $C_3(\Omega)$ (see [6], theorem 9). From the properties of the Lebesgue integral it follows from the assertion $V(\mathbf{x}) \in C_3(\Omega)$ that $V(\mathbf{x}) \in W^{(3)}_{m+1}(\Omega)$. Applying Sobolev's imbedding theorem [9] to $V(\mathbf{x})$ we find that $V(\mathbf{x}) \in C^{(2)}(\Omega)$; this proves the smoothness of the function $V(\mathbf{x})$.

Using Green's formula for the operator L [7]

$$\int_\Omega (L(V)\Theta - L^*(\Theta)V)\,d\mathbf{x} = -\int_{\Sigma_*}\left(\Theta\frac{\partial V}{\partial \nu} - V\frac{\partial \theta}{\partial \nu}\right)d\sigma - \int_\Sigma bV\Theta d\sigma$$

and substituting into it the expression (2.4) and also the equations $\Theta = 0,\ V = g$ on $\Sigma = \Sigma_3$, we obtain

$$\int_\Omega L(V)\Theta dx = 0 \tag{2.8}$$

Since Θ is an arbitrary smooth function not identically zero in Ω, and the operator $L(V)$ is continuous in Ω as a function of x, then it follows from (2.8) by the fundamental lemma of the calculus of variations that $L(V) = 0$ in Ω. We have thus proved the Property B. A proof of Property C is not necessary since its satisfaction was stipulated beforehand in the formulation of the boundary value problem (2.2). Property A is obvious since $h(0) = 0$ and, from the relation (2.1), it follows that $V(0) = 0$. This completes the proof of the theorem.

3. Thus, in studying a specific system of equations of a perturbed motion $\mathbf{x}^\cdot = f(\mathbf{x})$, it is necessary to construct the auxiliary system of equations $\mathbf{x}^\cdot = h(\mathbf{x})$ in accordance with the formula (1.3). For the auxiliary system we then pose the boundary value problem (2.2), where the number $g > 0$ is chosen arbitrarily and where r must be such that Ω will contain the phase space domain of interest for the system studied. The solution of the boundary value problem can be obtained by numerical methods with the aid of a computer [10]. Knowing the function $V(\mathbf{x})$, and having constructed the func-

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*) In the theorem used here there is the additional requirement that boundary points for Σ_3 be limit points for the internal points of Σ^0; certain conditions on the structure of the set G are also imposed. It can be shown that in case $\Sigma_3 = \Sigma$ these requirements are superfluous.

tions $P_+(x)$ and $P_-(x)$ in the domain Ω, we can determine the topological picture of the distribution of the sets H_1, H_2, H_3 in Ω, and, in accordance with the Theorems 1.1, 1.4 and 1.5, we can classify the stability of the trivial solution of the system (1.2). Lemma 1.1 makes it possible to carry over this classification to the system (1.1).

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ASYMPTOTIC METHOD FOR MULTI-DIMENSIONAL SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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We consider almost linear symmetric hyperbolic systems with constant coefficients in the linear part and with nonlinear terms containing a small parameter. The asymptotic method used here for construction of approximate solutions is based on the work of Bogoliubov and Mitropol'skii [1], and has been applied to systems with a single independent spatial variable [2, 3]. Along with a slow time we introduce slow coordinates. For the approximate solution we obtain, not an infinite system as in [4], but a finite system of almost linear partial differential equations with constant coefficients, a system which is simpler than the original one. We present an algorithm for obtaining approximate solutions. We also show